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Isolated Eigenvalue of a Random Matrix*

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A derivation is given of a property, noticed by Porter, of random real symmetric matrices. In an idealized physical problem, the random matrices were generated in a computer. If the elements of the matrices had a nonzero mean, then the matrices were found to have a statistically isolated eigenvalue. It is shown here that such an isolated eigenvalue exists for a large matrix if the mean value of the elements is a substantial fraction of the root mean square of the elements. The associated eigenvector has all its components close to equality. Counter examples are given, of matrices which are "statistical freaks," to show that the properties are statistical and not universal.

1. INTRODUCTION

MANY nuclear levels are known for which there is little hope of ever giving a precise discussion on the basis of some nuclear model. There is still considerable interest in the statistical properties of any group of levels. A well-defined quantity is the level density or the level spacing, and both show fluctuations which are of considerable physical interest. A conjecture by Wigner[1] concerns the distribution of spacings, using as a model the eigenvalues of a large random matrix. Porter and Rosenzweig[2] used a computer program to generate the elements of a sequence of random matrices. A second part of the program put each matrix into diagonal form for a study of the distribution of eigenvalues. Here and in a continuation of the computer studies[3] the predicted distributions were found. In particular, for a set of matrices whose elements were generated from a random distribution with mean zero, a histogram (with appropriate horizontal scale) of a number of eigenvalues in a given numerical range was approximately a semicircle.

When the portion of the program generating the matrix elements was altered so that the mean was no longer zero, it was found that the majority of the eigenvalues were still in the same histogram, but that there was one isolated eigenvalue.[4] Both in nuclear physics and in the study of superconductivity there is considerable interest in the existence of such an isolated eigenvalue. The state involved is no longer an anonymous member of a nondescript set, but the ground state of the system. It is the purpose of this paper, therefore, to establish the existence of such an isolated eigenvalue in the case of a biased random matrix, and to study the associated eigenvector.

In Sec. 2, it is shown that there is always an eigenvalue of a real symmetric matrix whose magnitude is greater than or equal to the magnitude of the mean of the sums of the elements in a row of the matrix. In Sec. 3, statistical conditions are established for such an eigenvalue to be isolated in magnitude. In Sec. 4 it is shown that, statistically, the associated eigenvector of the isolated eigenvalue has most of its components close to the mean component.

2. LOWER LIMIT ON EIGENVALUE MAGNITUDE

We shall treat a real symmetric $N\times N$ matrix which has $\frac{1}{2}N(N+1)$ independent elements a_{ij}. There are N eigenvalues λ_k, each corresponding to an eigenvector of N elements

$$\mathbf{U}_k=(U_{ki}). \tag{2.1}$$

It is assumed that the $\mathbf{U}_k$ are orthonormal. Then there is a relation

$$a_{ij}=\textstyle\sum_k \lambda_k U_{ki}U_{kj}. \tag{2.2}$$

If all rows (and hence all columns) of the matrix have the same sum, μ, then there is an eigenvalue μ with the obvious eigenvector $U_{\mu i}=N^{-1/2}$ for all i. The matrix $b_{ij}=a_{ij}-\mu/N$ has the same set of eigenvectors, and the only changes are that in Eq. (2.2) zero replaces the eigenvalue μ, and the mean value of the matrix elements is zero.

Even if there is not such an exact property, there is a mean μ of the sums of rows. For the remainder of the paper we shall assume, without loss of generality, that μ is positive. Then, unless μ is an eigenvalue, there is always at least one eigenvalue greater than μ, and at least one less than μ. A measure of the failure of any quantity μ' as an eigenvalue with the simple vector $\mathbf{x}$ equal to the eigenvector $\mathbf{U}_\mu$ above would be

$$F^2=(Ax-\mu' x)^2 \tag{2.3}$$

$$=\frac{1}{N}\sum_j(\sum_i a_{ij}-\mu')^2. \tag{2.4}$$

Now $\mathbf{x}$ can be expressed as a linear combination of a set of orthonormal eigenvectors of the matrix.

$$\mathbf{x}=\textstyle\sum_k \alpha_k\mathbf{U}_k, \tag{2.5}$$

where

$$\alpha_k=\textstyle\sum_i N^{-1/2}U_{ki}. \tag{2.6}$$

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The expression (2.3) can then be rewritten.

$$F^2=\sum_k[(A-\mu' I)\alpha_k U_k]^2, \tag{2.7}$$

$$=\sum_k[(\lambda_k-\mu')I\alpha_k U_k]^2, \tag{2.8}$$

$$=\sum_k[(\lambda_k-\mu')^2\alpha_k^2]. \tag{2.9}$$

The value of μ' which makes F^2 a minimum is found by differentiation.

$$0=(\partial/\partial\mu')\sum_k \alpha_k^2(\lambda_k-\mu')^2 \tag{2.10}$$

$$=2\sum_k \alpha_k^2(\mu'-\lambda_k), \tag{2.11}$$

i.e.,

$$\mu'=\sum_k \alpha_k^2\lambda_k. \tag{2.12}$$

From Eq. (2.11) it is seen that at least one member of the λ_k is greater than the value of μ' satisfying Eq. (2.12) and at least one member is less. Using the expression (2.4), equivalent to expression (2.8), another form of (2.11) is

$$0=2\frac{1}{N}\sum_j(\mu'-\sum a_{ij}), \tag{2.13}$$

$$\mu'=\frac{1}{N}\sum_{ij} a_{ij}. \tag{2.14}$$

Hence, $\mu'=\mu$ to satisfy Eq. (2.12), and the range of the eigenvalues includes μ.

3. ISOLATION OF AN EIGENVALUE

For a sufficiently high power r, the sum of the rth powers of the eigenvalues will approximate the rth power of the eigenvalue largest in absolute magnitude, and it may be possible to obtain an upper limit on the magnitude of the eigenvalue second largest in absolute magnitude. Since the eigenvalue whose isolation we seek to prove is of the order $N\langle a_{ij}\rangle_{\rm av}(=N\bar{a})$, and the sum of the squares of the eigenvalues is $\sum a_{ij}^2=N^2\langle a_{ij}^2\rangle_{\rm av}$, subtraction of the square of the special eigenvalue does not exhibit the isolation in general.

We therefore consider the sum of the fourth powers of the eigenvalues. This is the trace of the fourth power of the matrix.

$$\sum_k \lambda_k^4=\sum_{i,j,k,l} a_{ij}a_{jk}a_{kl}a_{li}. \tag{3.1}$$

If it is assumed that the values of the elements are uncorrelated, the mean value of the right-hand side may be written in terms of the means of powers of elements of the matrix. Elements on the diagonal are separated for interest and are designated by the symbol b.

$$\begin{aligned}\sum_k \lambda_k^4=&N(N-1)(N-2)(N-3)\bar{a}^4\\&+2N(N-1)(N-2)\{\langle a^2\rangle_{\rm av}^2+2\bar{a}^3\bar{b}\}\\&N(N-1)\{2\langle a^2\rangle_{\rm av}\bar{b}^2+\langle a^4\rangle_{\rm av}\\&\quad+4\langle a^2\rangle_{\rm av}\langle b^2\rangle_{\rm av}\}+N\langle b^4\rangle_{\rm av}\end{aligned} \tag{3.2}$$

$$\begin{aligned}=&N^4(\bar{a}^4)+N^3\{-6\bar{a}^4+2\langle a^2\rangle_{\rm av}^2+4\bar{a}^3\bar{b}\}\\&+N^2\{11\bar{a}^4-6\langle a^2\rangle_{\rm av}^2-12\bar{a}^3\bar{b}+2\langle a^2\rangle_{\rm av}\bar{b}^2\\&+\langle a^4\rangle_{\rm av}+4\langle a^2\rangle_{\rm av}\langle b^2\rangle_{\rm av}\}\\&+N\{-6\bar{a}^4+4\langle a^2\rangle_{\rm av}^2+8\bar{a}^3\bar{b}-2\langle a^2\rangle_{\rm av}\bar{b}^2\\&\quad-\langle a^4\rangle_{\rm av}+\langle b^4\rangle_{\rm av}-4\langle a^2\rangle_{\rm av}\langle b^2\rangle_{\rm av}\}.\end{aligned} \tag{3.3}$$

For large enough N and positive $\bar{a}$, this sum is closely equal to the fourth power of the special root which is greater than $N\bar{a}$. The root will then lie in the range

$$N\bar{a}\leqq\lambda_s\leqq N\bar{a}[1+\tfrac{1}{4}N^{-1}\{2\langle a^2\rangle_{\rm av}^2/\bar{a}^4+4\bar{b}/\bar{a}-6\}+O(N^{-2})]. \tag{3.4}$$

Since there is an eigenvalue larger than $N\bar{a}$, or more strictly than $(N-1)\bar{a}+\bar{b}$, the sum of the fourth powers of the previous eigenvalues, and hence the magnitude of the largest remaining eigenvalue will be less than some fixed magnitude multiplied by $N^{3/4}$, and isolation of the eigenvalue can be achieved by taking a large enough value of N. The existence and isolation of a large eigenvalue are statistical properties only. Consider the "statistical freak" $2N\times 2N$ matrix which is the direct product of an $N\times N$ matrix consisting only of ones and the matrix

$$\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

This matrix has only two nonzero eigenvalues $\pm\sqrt{2}N$. The $4N\times 4N$ matrix which is the direct product of the unit matrix of order two and the product just described has only four nonzero eigenvalues which are degenerate in pairs at $\pm\sqrt{2}N$. Statistically both examples would be expected to have an isolated eigenvalue near N.

4. ASSOCIATED EIGENVECTOR OF THE ISOLATED EIGENVALUE

It now remains to show that the "special" eigenvector is closely related (statistically) to the vector whose elements are all $N^{-1/2}$.

Combining Eqs. (2.4) and (2.9), and substituting Eq. (2.14) for the best value of the trial eigenvalue with this vector gives

$$\sum_k(\lambda_k-N\bar{a})^2\alpha_k^2\approx N\langle a^2\rangle_{\rm av}-N\bar{a}^2. \tag{4.1}$$

By making N large enough, all except one member of the set λ_k can be made less than $\frac{1}{2}N\bar{a}$ (positive). Then

$$\sum_{k\neq s}{}' \tfrac{1}{4}N^2\bar{a}^2\alpha_k^2\leqq N(\langle a^2\rangle_{\rm av}-\bar{a}^2). \tag{4.2}$$

The prime on the summation indicates that one term has been omitted. For the special eigenvalue λ_s

$$\alpha_s^2=1-\sum_{k\neq s}{}'\alpha_k^2, \tag{4.3}$$

$$1\geqq\alpha_s\geqq 1-\frac{2}{N}(\langle a^2\rangle_{\rm av}/\bar{a}^2-1). \tag{4.4}$$

Now

$$\alpha_s = \mathbf{x}\cdot\mathbf{U}_s = \sum_i N^{-1/2} U_{si}. \tag{4.5}$$

Then, if

$$U_{si} = N^{-1/2}(1-\nu_{si}), \tag{4.6}$$

$$0 \leqq \sum_i \nu_{si}^2 = 2\sum_i \nu_{si} \leqq 4(\langle a^2\rangle_{\text{av}}/\bar{a}^2 - 1). \tag{4.7}$$

The right-hand side of Eq. (4.7) is independent of N, and therefore, as N increases the fraction of the ν_{si} which deviate significantly from zero will shrink.

Note that the property shown is again statistical. Consider the "statistical freak" $2N\times 2N$ matrix which is the direct product of an $N\times N$ matrix with all elements one and the two-by-two matrix

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The product matrix has the eigenvalue $+2N$ and all other eigenvalues are zero. The associated eigenvector of the isolated eigenvalue has the first N elements equal to $(2N)^{-1/2}$ and the rest equal to $-(2N)^{-1/2}$. The vector with all elements $(2N)^{-1/2}$ is also an eigenvector in this case.

Note also that if in Eq. (2.2) the value zero is substituted instead of the special isolated eigenvalue, then the remaining matrix will closely resemble the matrix originally postulated, except in having mean close to zero for its elements. This suggests, but no proof is offered, that the remaining eigenvalues of the original matrix will be distributed in accordance with the properties already found for a matrix of random elements with zero mean.[2,4]

Note added in proof. In the first paper of Ref. 1, the semicircular distribution of eigenvalues was shown to follow from a certain very restricted distribution of matrix elements. In the second paper the conditions to be obeyed by the matrix elements were very considerably relaxed. The demonstration was unchanged. One restriction still required was an identically zero mean for the matrix elements. A simple extension of the argument can be used to relax this restriction and allow a small nonzero mean. The meaning to be attached to 'small' is that $N^2\bar{a}^2 \ll N\langle a^2\rangle_{\text{av}}$.

In the case of the matrix obtained by substituting $\lambda_s=0$ in Eq. (2.2), the elements have nonzero mean of sign opposite to λ_s. The magnitude can be seen from Eq. (3.6) to be of order N^{-1} times a constant. The remaining moments about the mean are essentially unaltered. Thus the relaxed condition is easily fulfilled. All other conditions in Ref. 1 are met by the new matrix if they applied to the original. The conclusion can be stated for the set of general random matrices whose elements obey all the conditions specified by Wigner except that the mean is nonzero. Any one of the set of matrices will exhibit statistically one isolated eigenvalue, associated with an eigenvector which is close having all its elements equal, and the remainder of the eigenvalues will lie in a semicircular histogram.

5. CONCLUSIONS

It has been shown that for a real symmetric matrix, of sufficiently high order, whose elements have nonzero mean, there is an eigenvalue which is statistically close to the mean value of the sum of a row of the matrix. The associated eigenvector has its elements statistically close to all being equal. The aim of further study should be to treat the case which appears more commonly in physical problems, where the diagonal elements form a monotonic sequence, and the nonzero off-diagonal elements are randomly distributed about a small nonzero mean.

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